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Inhomogeneous Diophantine approximation on curves and Hausdorff dimension

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Abstract

The goal of this paper is to develop a coherent theory for inhomogeneous Diophantine approximation on curves in $\mathbb{R}^n$ akin to the well established homogeneous theory. More specifically, the measure theoretic results obtained generalize the fundamental homogeneous theorems of R.C. Baker (1978) [2], Dodson, Dickinson (2000) [18] and Beresnevich, Bernik, Kleinbock, Margulis (2002) [8]. In the case of planar curves, the complete Hausdorff dimension theory is developed.

Keywords: Diophantine approximation; Lebesgues measure; Hausdorff dimension; Non-degenerate curve; Khintchine theorem

1. Introduction

Throughout $\psi : \mathbb{R}^+ \to \mathbb{R}^+ := (0, +\infty)$ denotes a decreasing function and will be referred to as an *approximation function*. Let $\mathbf{f} = (f_1, \ldots, f_n) : I \to \mathbb{R}^n$ be a $C^{(n)}$ map defined on an interval $I \subset \mathbb{R}$ and $\lambda : I \to \mathbb{R}$ be a function. For reasons that will soon be apparent, the function λ will be referred to as an *inhomogeneous function*. Let $A_n(\psi, \lambda)$ be the set of $x \in I$ such that the inequality

$$\|\mathbf{a} \cdot \mathbf{f}(x) + \lambda(x)\| < \psi(|\mathbf{a}|) \tag{1}$$

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holds for infinitely many $\mathbf{a} \in \mathbb{Z}^n \setminus \{0\}$, where $\|\cdot\|$ denotes the distance to the nearest integer, $|\mathbf{a}| := \max\{|a_1|, \ldots, |a_n|\}$ and $\mathbf{a} \cdot \mathbf{b}$ stands for the standard inner product of vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{Z}^n$. In the special case when $\psi(h) = \psi_v(h) := h^{-v}$ for some fixed positive v we will denote $A_n(\psi_v, \lambda)$ by $A_n(v, \lambda)$. Furthermore, in the case when the inhomogeneous function λ is identically zero we write $A_n(\psi)$ for $A_n(\psi, \lambda)$ and $A_n(v)$ for $A_n(v, \lambda)$.

By definition, $A_n(\psi, \lambda)$ is the set of $x \in I$ such that the corresponding point $\mathbf{f}(x)$ lying on the curve

$$\mathcal{C} := \{(f_1(x), f_2(x), \ldots, f_n(x)) \colon x \in I\} \subset \mathbb{R}^n \tag{2}$$

satisfies the Diophantine condition arising from (1). Within the homogeneous setup ($\lambda \equiv 0$), investigating the measure theoretic properties of $A_n(\psi)$ dates back to 1932 and a famous problem of Mahler [24]. The problem states that when (2) is the Veronese curve given by $(x, x^2, \ldots, x^n)$ then $A_n(v)$ is of Lebesgue measure zero whenever $v > n$. Mahler's problem was eventually settled by Sprindžuk [28,29] in 1967 and subsequently Schmidt [27] extended the result to the case of arbitrary planar curves (i.e. $n = 2$) with non-vanishing curvature. These two major results led to what is currently known as the (homogeneous) theory of Diophantine approximation on manifolds [14].

Diophantine approximation on manifolds has been an extremely active research area over the past 10 years or so. Rather than describe the activity in detail, we refer the reader to research articles [3,4,8,10,15,23,30] and the surveys [6,22,25]. Nevertheless, it is worth singling out the pioneering work of Kleinbock and Margulis [23] in which the fundamental Baker–Sprindžuk conjecture is established. This has undoubtedly acted as the catalyst to the works cited above which together constitute a coherent homogeneous theory for Diophantine approximation on manifolds. The situation for the inhomogeneous theory is quite different. Indeed, the inhomogeneous analogue of the Baker–Sprindžuk conjecture [11,12] has only just been established in 2008.

The aim of this paper is to develop a coherent theory for inhomogeneous Diophantine approximation on curves akin to the well established homogeneous theory. More precisely, Hausdorff measure theoretic statements for the sets $A_n(\psi, \lambda)$ are obtained. In particular, a complete metric theory is established in the case of planar curves ($n = 2$). In short, the results constitute the first precise and general statements in the theory of inhomogeneous Diophantine approximation on manifolds.

1.1. Main results and corollaries

Before we proceed with the statement of the results, we introduce some useful notation and recall some standard definitions.

The curve $\mathcal{C}$ given by (2) is called *non-degenerate* at $x \in I$ if the Wronskian

$$w(f_1', \ldots, f_n')(x) := \det(f_i^{(j)}(x))_{1 \leqslant i,j \leqslant n}$$

does not vanish. We say that $\mathcal{C}$ is *non-degenerate* if it is non-degenerate at almost every point $x \in I$. Given a set $X \subset \mathbb{R}^n$ and a real number $s > 0$, $\mathcal{H}^s(X)$ will denote the s-dimensional Hausdorff measure of X and $\dim X$ will denote the Hausdorff dimension of X. The latter is defined to be the infimum over s such that $\mathcal{H}^s(X) = 0$. For the formal definitions and properties of Hausdorff measure and dimension see [21].

1.1.1. Lower bounds

Our first result enables us to deduce lower bounds for $\dim A_n(v,\lambda)$ and represents an inhomogeneous version of the homogeneous theorem established by Dodson and Dickinson [18]. Furthermore, even within the homogeneous setup the result is stronger — it deals with Hausdorff measure rather than just dimension.

Theorem 1. *Let* $\mathbf{f} \in C^{(n)}(I)$, ψ *be an approximation function and* $\lambda \in C^{(2)}(I)$. *Assume that* $w(f_1', \dots, f_n')(x) \neq 0$ *for all* $x \in I$. *Then for any* $0 < s \leqslant 1$

$$\mathcal{H}^s\big(A_n(\psi,\lambda)\big) = \mathcal{H}^s(I) \quad \text{if} \sum_{q=1}^{\infty} \left(\frac{\psi(q)}{q}\right)^s \cdot q^n = \infty. \tag{3}$$

Note that whenever the sum in (3) diverges, the theorem implies that $\mathcal{H}^s(A_n(\psi,\lambda)) > 0$. In turn, it follows from the definition of Hausdorff dimension that $\dim(A_n(\psi,\lambda)) \geqslant s$. In particular, it is easily verified that the sum in (3) diverges whenever $s < (n+1)/(1+\tau_\psi)$, where

$$\tau_\psi := \liminf_{q\to\infty} \frac{-\log \psi(q)}{\log q}$$

is the lower order of $1/\psi$ at infinity. Thus, Theorem 1 readily gives the following inhomogeneous version of the Dodson–Dickinson lower bound [18] for non-degenerate curves.

Corollary 1. *Let* $\mathbf{f}$, ψ *and* λ *be as in Theorem* 1 *with* $\tau_\psi \geqslant n$. *Then*

$$\dim A_n(\psi,\lambda) \geqslant \frac{n+1}{\tau_\psi + 1}. \tag{4}$$

Note that in the case of $\psi(q) = q^{-v}$ we have that $\tau_\psi = v$ as one would expect. In the case of λ being a constant function and $\psi(q) = q^{-v}$, the above corollary has previously been obtained by the author in [1]. Bugeaud [16] has established (4) within the context of approximation by algebraic integers; i.e. in the case that $\mathcal{C}$ is the Veronese curve $(x^n, \dots, x)$ and $\lambda : x \to x^{n+1}$.

In the case $s = 1$, the s-dimensional Hausdorff measure $\mathcal{H}^s$ is simply one-dimensional Lebesgue measure on the real line $\mathbb{R}$. Thus, Theorem 1 trivially gives rise to a complete inhomogeneous analogue of the theorem of Beresnevich, Bernik, Kleinbock and Margulis [8] in the case of non-degenerate curves.

Corollary 2. *Let* $\mathbf{f}$, ψ *and* λ *be as in Theorem* 1. *Furthermore, suppose that the associated curve given by* (2) *is non-degenerate. Then*

$$\big|A_n(\psi,\lambda)\big| = |I| \quad \text{if} \sum_{h=1}^{\infty} h^{n-1}\psi(h) = \infty.$$

Here and elsewhere $|X|$ will stand for the Lebesgue measure of a measurable subset X of $\mathbb{R}$.

1.1.2. Upper bounds

It is believed that the lower bound for $\dim A_n(\psi,\lambda)$ given in Corollary 1 is sharp. Establishing that this is the case, represents a challenging problem and in general is open even in the homogeneous setup — it has only been verified in some special cases [2,7,13,19]. In particular, Baker [2] has settled the problem for planar curves within the homogeneous setup. To the best of our knowledge, nothing seems to be known in the inhomogeneous case. The following result, which is an inhomogeneous generalization of Baker's theorem, gives a complete theory for planar curves in the inhomogeneous case.

Theorem 2. *Let $\psi:\mathbb{R}^+\to\mathbb{R}^+$ be an approximation function with $\tau_\psi\geqslant 2$. Let $f_1, f_2, \lambda\in C^{(2)}(I)$ be such that the associated curve $\mathcal{C}$ given by $(2)_{n=2}$ is non-degenerate everywhere except possibly on a set of Hausdorff dimension less than $\frac{3}{\tau_\psi+1}$. Then*

$$\dim A_2(\psi,\lambda)=\frac{3}{\tau_\psi+1}.$$

2. Lower bounds: proof of Theorem 1

The proof of Theorem 1 will rely on the ubiquitous systems technique as developed in [9]. Essentially, the notion of a ubiquitous system represents a convenient way of describing the 'uniform' distribution of the naturally arising points (and more generally sets) from a given Diophantine approximation inequality/problem — see [9,20].

2.1. Ubiquitous systems in $\mathbb{R}$

For the sake of simplicity, we introduce a restricted notion of a ubiquitous system, which is more than adequate for the applications we have in mind.

Let I_0 be an interval in $\mathbb{R}$ and $\mathcal{P}=(P_\alpha)_{\alpha\in J}$ be a family of resonant points P_α of I_0 indexed by an infinite set J. Next, let $\beta: J\to\mathbb{R}^+:\alpha\mapsto\beta_\alpha$ be a positive function on J. Thus the function β attaches a 'weight' β_α to the resonant point P_α. Assume that for every $t\in\mathbb{N}$ the set $J_t=\{\alpha\in J:\ \beta_\alpha\leqslant 2^t\}$ is finite.

Throughout, $\rho:\mathbb{R}^+\to\mathbb{R}^+$ will denote a function such that $\rho(t)\to 0$ as $t\to\infty$ and it will be referred to as a *ubiquitous function*. Also, $B(x,r)$ will denote the ball (or rather the interval) centered at x with radius r.

Definition 1. Suppose that there exists a ubiquitous function ρ and an absolute constant $k>0$ such that for any interval $I\subseteq I_0$

$$\liminf_{t\to\infty}\left|\bigcup_{\alpha\in J_t}B\big(P_\alpha,\rho\big(2^t\big)\big)\cap I\right|\geqslant k|I|.$$

Then the system $(\mathcal{P},\beta)$ is called locally ubiquitous in I_0 with respect to ρ.

Let $(\mathcal{P},\beta)$ be a ubiquitous system with respect to ρ and Ψ be an approximation function. Let $\Lambda(\mathcal{P},\beta,\Psi)$ be the set of points $\xi\in\mathbb{R}$ such that the inequality

$$|\xi-P_\alpha|<\Psi(\beta_\alpha)$$

holds for infinitely many $\alpha \in J$. We will be making use of the following lemma, which is an easy consequence of Corollary 2 (in the case $s = 1$) and Corollary 4 (in the case $s < 1$) from [9].

Lemma 1. *Let ψ be an approximation function and $(\mathcal{P}, \beta)$ be a locally ubiquitous system with respect to ρ. Suppose there exists a real number $\lambda \in (0, 1)$ such that $\rho(2^{t+1}) < \lambda\, \rho(2^t)$ for all $n \in \mathbb{N}$. Then,*

$$\mathcal{H}^s\big(\Lambda(\mathcal{P}, \beta, \Psi)\big) = \mathcal{H}^s(I_0) \quad \textit{if} \sum_{t=1}^{\infty} \frac{\Psi(2^t)^s}{\rho(2^t)} = \infty.$$

2.2. Reduction of Theorem 1 to a ubiquity statement

First some notation. Let $\mathbf{f} = (f_1, \ldots, f_n)$ be as in Theorem 1 and denote by $\mathcal{F}_n$ the set of all functions

$$a_0 + a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x)$$

where $a_0, \ldots, a_n$ are integer coefficients, not all zero. Given a function $F \in \mathcal{F}_n$, the *height* $H(F)$ of F is defined as

$$H(F) := \max\big\{|a_1|, \ldots, |a_n|\big\}.$$

For $H > 1$, let $\mathcal{F}_n(H)$ denote the subclass of $\mathcal{F}_n$ given by

$$\mathcal{F}_n(H) = \big\{F \in \mathcal{F}_n\colon\ H(F) \leqslant H\big\}.$$

Given an inhomogeneous function λ, let $R_\lambda = \{\alpha \in I\colon\ \exists F \in \mathcal{F}_n,\ F(\alpha) + \lambda(\alpha) = 0\}$. Then, for $\alpha \in R_\lambda$ the quantity $H(\alpha) := \min\{H(F) \mid F \in \mathcal{F}_n,\ F(\alpha) + \lambda(\alpha) = 0\}$ will be referred to as the height of α.

To illustrate the above notions, consider the following concrete example. Let the functions $f_i(x) = x^i$ be powers of x. Then $\mathcal{F}_n$ is simply the set of all non-zero integral polynomials of degree at most n. Furthermore, if λ is identically zero, then R_λ is simply the set of algebraic numbers in I of degree at most n. On the other hand, if $\lambda(x) = x^{n+1}$ then R_λ is simply the set of algebraic integers in I of degree exactly $n + 1$.

The key to establishing Theorem 1 is the following ubiquity statement.

Proposition 1. *The system $(R_\lambda, H(\alpha))$ is locally ubiquitous in I with respect to $\rho(q) = q^{-n-1}$.*

We postpone the proof of Proposition 1 to the next section. We now establish Theorem 1 modulo the proposition. Note that without loss of generality we can assume that I is a closed interval. Then, since the functions $f_i^{(j)}$ and $\lambda^{(k)}$, $0 \leqslant j \leqslant n$, $1 \leqslant i \leqslant n$, $0 \leqslant k \leqslant 2$ are continuous we have that

$$\forall x \in I \quad \big|f_i^{(j)}(x)\big| \leqslant C, \qquad \big|\lambda^{(k)}(x)\big| \leqslant C \tag{5}$$

for some absolute constant C. Therefore we get

$$|F'(x)| \leqslant nCH(F) := MH(F).$$

Let $\alpha \in R_\lambda$. Then, by definition there exists a function $F \in \mathcal{F}_n$ such that $F(\alpha) + \lambda(\alpha) = 0$. Consider the interval

$$J := \bigl(\alpha - (2M)^{-1}H(F)^{-1}\psi\bigl(H(F)\bigr), \alpha + (2M)^{-1}H(F)^{-1}\psi\bigl(H(F)\bigr)\bigr).$$

For any $x \in J \cap I$, we have that

$$|F'(x) + \lambda'(x)| \leqslant MH(F) + C \leqslant 2MH(F). \tag{6}$$

The latter inequality holds for all sufficiently large $H(F)$. Using the Mean Value Theorem, we obtain that for any $x \in J \cap I$:

$$F(x) + \lambda(x) = F(\alpha) + \lambda(\alpha) + \bigl(F'(x_2) + \lambda'(x_2)\bigr)(x - \alpha),$$

where x_2 lies between α and x. By (6), we get that $|F(x) + \lambda(x)| \leqslant \psi(H(F))$ and so it follows that

$$\Lambda\bigl(R_\lambda, H(\alpha), (2M)^{-1}H(F)^{-1}\psi\bigl(H(F)\bigr)\bigr) \subset A_n(\psi, \lambda). \tag{7}$$

In view of the divergent sum condition in Eq. (3), we have that

$$\sum_{t=1}^{\infty} \frac{((2M)^{-1}2^{-t}\psi(2^t))^s}{2^{-t(n+1)}} \asymp \sum_{t=1}^{\infty} 2^{t(n+1)}\left(\frac{\psi(2^t)}{2^t}\right)^s \geqslant \sum_{h=1}^{\infty} h^n\left(\frac{\psi(h)}{h}\right)^s = \infty.$$

Thus, Lemma 1 implies that

$$\mathcal{H}^s\bigl(\Lambda\bigl(R_\lambda, H(\alpha), (2M)^{-1}H(F)^{-1}\psi\bigl(H(F)\bigr)\bigr)\bigr) = \mathcal{H}^s(I).$$

This together with (7) implies that

$$\mathcal{H}^s\bigl(A_n(\psi, \lambda)\bigr) = \mathcal{H}^s(I).$$

Modulo establishing Proposition 1, this completes the proof of Theorem 1.

2.3. *Proof of Proposition 1*

Without loss of generality we can assume that $f_1(x) = x$ as otherwise we can use the Inverse Function Theorem to change variables and ensure the condition. Let $\Phi(Q, \delta)$ denote the set of $x \in I$ such that

$$|F(x)| < \delta Q^{-n} \tag{8}$$

for some $F \in \mathcal{F}_n(Q)$. We shall make use of the following lemma regarding the measure of $\Phi(Q, \delta)$.

Lemma 2. *There is an absolute constant $\delta > 0$ with the following property: for any $x_0 \in I$ there is a neighborhood $I_0 \subset I$ of x_0 such that for any interval $J \subset I_0$ there exists a sufficiently large $Q_1 > 0$ such that for all $Q > Q_1$ we have $|\Phi(Q,\delta) \cap J| < |J|/2$.*

This lemma is a consequence of Theorem 2.1 in [8]. Since I is compact, it is easy to see that I_0 can be taken to be I.

Take Q_1 and δ from Lemma 2. Define $C_1 := \delta^{\frac{1}{n+1}}$ and fix some number $Q > \frac{1}{C_1} Q_1$. Let $\xi \in I \backslash \Phi(C_1 Q, \delta)$. The goal is to show that we can find $\alpha \in R_\lambda$ such that

$$H(\alpha) \leqslant K_1 Q \quad \text{and} \quad |\xi - \alpha| \leqslant K_2 Q^{-n-1} \tag{9}$$

where the constants K_1 and K_2 are independent from both Q and J. It would immediately follow that for $Q > \frac{1}{C_1} Q_1$,

$$\begin{aligned} \frac{|J|}{2} \leqslant \left| J \backslash \Phi(C_1 Q, \delta) \right| &\leqslant \left| \bigcup_{H(\alpha) \leqslant K_1 Q} B\big(\alpha, K_2 Q^{-n-1}\big) \cap J \right| \\ &= \left| \bigcup_{H(\alpha) \leqslant Q'} B\big(\alpha, K_2 K_1^{n+1} (Q')^{-n-1}\big) \cap J \right| \end{aligned}$$

where $Q' = K_1 Q$. Therefore

$$\left| \bigcup_{H(\alpha) \leqslant Q} B\big(\alpha, Q^{-n-1}\big) \cap J \right| \geqslant \frac{|J|}{2 K_2 K_1^{n+1}}. \tag{10}$$

Taking $Q = 2^t$ and setting $\rho(H) := H^{-n-1}$, inequality (10) implies that $(R_\lambda, H(\alpha))$ is locally ubiquitous in I with respect to ρ — the statement of Proposition 1.

We now proceed to establish (9). Consider the system of inequalities

$$\begin{cases} \left| a_n f_n(\xi) + \cdots + a_1 f_1(\xi) + a_0 \right| < Q^{-n}; \\ |a_1|, |a_2|, \ldots, |a_n| \leqslant Q. \end{cases} \tag{11}$$

It defines a convex body in $\mathbb{R}^{n+1}$ symmetric about the origin. Consider its successive minima $\tau_1, \ldots, \tau_{n+1}$. By definition, $\tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_{n+1}$. Note that $\tau_1 > C_1$. Indeed, otherwise we would have $H \leqslant C_1 Q$ and

$$\left| a_n f_n(\xi) + \cdots + a_0 \right| \leqslant C_1 Q^{-n} = C_1^{n+1} (C_1 Q)^{-n} \leqslant \delta H^{-n},$$

a contradiction. By Minkowski's theorem on successive minima [17], $\tau_1 \cdots \tau_{n+1} \leqslant 1$. Thus, we obtain the bound

$$\tau_{n+1} \leqslant (\tau_1 \cdot \tau_2 \cdots \tau_n)^{-1} < C_1^{-n} = C_2$$

where C_2 is an absolute constant depending only on Q. Finally, by the definition of τ_{n+1}, we obtain the set of $n+1$ linearly independent functions $F_j(X) = a_n^{(j)} f_n(X) + \cdots + a_1^{(j)} X + a_0^{(j)}$, $1 \leqslant j \leqslant n+1$ with integer coefficients $a_i^{(j)}$ such that

$$\begin{cases} \left|F_j(\xi)\right| \leqslant C_2 Q^{-n}; \\ \left|a_i^{(j)}\right| \leqslant C_2 Q, \quad i = \overline{1,n}. \end{cases} \tag{12}$$

Now consider the following system of linear equations

$$\begin{cases} \theta_1 F_1(\xi) + \cdots + \theta_{n+1} F_{n+1}(\xi) + \lambda(\xi) = 0; \\ \theta_1 F_1'(\xi) + \cdots + \theta_{n+1} F_{n+1}'(\xi) + \lambda'(\xi) = Q + \sum_{i=1}^{n} \left|F_i'(\xi)\right|; \\ \theta_1 a_j^{(1)} + \cdots + \theta_{n+1} a_j^{(n+1)} = 0, \quad 2 \leqslant j \leqslant n. \end{cases} \tag{13}$$

We transform this system in the following manner. Consider the left hand side of the second equation in (13). It equals to

$$\sum_{i=1}^{n+1} \theta_i \sum_{j=1}^{n} a_j^{(i)} f_j'(\xi) = \sum_{j=1}^{n} f_j'(\xi) \sum_{i=1}^{n+1} \theta_i a_j^{(i)}.$$

According to (13) we get that for $2 \leqslant j \leqslant n$ all terms in summation are equal to zero. Also since $f_1(x) \equiv x$ then the second row in (13) will have the form $\theta_1 a_1^{(1)} + \cdots + \theta_{n+1} a_1^{(n+1)}$. Similarly we can transform the first row to the form $\theta_1 a_0^{(1)} + \cdots + \theta_{n+1} a_0^{(n+1)}$. Since the matrix $(a_i^{(j)})$, $0 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n+1$ is non-degenerate, the system (13) has a unique solution $\theta_1, \dots, \theta_{n+1}$. Choose integers $t_1, t_2, \dots, t_n$ such that $|t_i - \theta_i| < 1$, $1 \leqslant i \leqslant n+1$. Consider the function

$$\begin{aligned} F(X) &= t_1 F_1(X) + \cdots + t_{n+1} F_{n+1}(X) \\ &= x_n f_n(X) + \cdots + x_1 X + x_0, \end{aligned}$$

where $x_i = t_1 a_i^{(1)} + \cdots + t_{n+1} a_i^{(n+1)}$. By (12) and the first equation in (13), we obtain

$$\left|F(\xi) + \lambda(\xi)\right| < (n+1) C_2 Q^{-n} = C_3 Q^{-n}.$$

Further, by the second equation in (13), we obtain $|F'(\xi) + \lambda'(\xi)| > Q$ and

$$\begin{aligned} \left|F'(\xi) + \lambda'(\xi)\right| &< Q + 2\sum_{i=1}^{n+1} \left|F_i'(\xi)\right| \leqslant Q + 2(n+1)\big((n+1)C_2 \cdot CQ\big) \\ &= \big(1 + 2(n+1)^2 C_2 \cdot C\big) Q = C_4 Q. \end{aligned}$$

Now consider the coefficients x_i. They are obviously integers. By the third equation in (13), we get $|x_m| \leqslant (n+1) C_2 Q = C_5 Q$ for all $m \geqslant 2$. The bounds for x_0 and x_1 are given by

$$\begin{aligned} |x_1| &\leqslant \left|F'(\xi)\right| + \left|\lambda'(\xi)\right| + \sum_{i=2}^{n} \left|x_i f_i'(\xi)\right| \\ &\leqslant C_4 Q + (n-1)(n+1) C_2 Q \cdot C + C \leqslant C_6 Q \end{aligned}$$

and

$$|x_0| \leqslant \left|F(\xi)\right| + \left|\lambda(\xi)\right| + \sum_{i=1}^{n}\left|x_i f_i(\xi)\right|$$
$$\leqslant C_3 Q^{-n} + (n-1)(n+1)C_2 \cdot CQ + C_6 CQ + C \leqslant C_7 Q$$

for sufficiently large Q. Thus, for every $\xi \in I \backslash \Phi(C_1 Q, \delta)$ there exists $F(x) \in \mathcal{F}_n$ such that

$$\begin{cases} \left|F(\xi) + \lambda(\xi)\right| \leqslant C_3 Q^{-n}; \\ Q \leqslant \left|F'(\xi) + \lambda'(\xi)\right| \leqslant C_4 Q; \\ \left|H(F)\right| \leqslant \max(C_5, C_6, C_7) Q. \end{cases} \tag{14}$$

It is easy to check that $\max\{C_5, C_6, C_7\} = C_7$. Hence, $|H(F)| \leqslant C_7 Q$ or equivalently $F \in \mathcal{F}_n(C_7 Q)$.

The next goal is to show that the function $F(x) + \lambda(x)$ constructed above has a root α satisfying conditions (9).

Lemma 3. *Let $\sigma(F)$ be the set of all $x \in I$ satisfying the following system of inequalities:*

$$\begin{cases} \left|F(x) + \lambda(x)\right| \leqslant C_3 Q^{-n}; \\ Q \leqslant \left|F'(x) + \lambda'(x)\right| \leqslant C_4 Q, \end{cases}$$

where $F \in \mathcal{F}_n(C_7 Q)$. Let Q satisfy the condition

$$(n \cdot C \cdot C_7 Q + C) \cdot 2C_3 Q^{-n-1} + C \leqslant \frac{1}{2} Q.$$

Then for all $x_0 \in \sigma(F) \cap [a + 2C_3 Q^{-n-1}, b - 2C_3 Q^{-n-1}]$ there exists a number $\alpha \in (x_0 - 2C_3 Q^{-n-1}, x_0 + 2C_3 Q^{-n-1})$ such that $F(\alpha) + \lambda(\alpha) = 0$.

Proof. Take an arbitrary $x \in (x_0 - 2C_3 Q^{-n-1}, x_0 + 2C_3 Q^{-n-1})$. By the Mean Value Theorem,

$$F'(x) + \lambda'(x) = F'(x_0) + \lambda'(x_0) + \left(F''(x_1) + \lambda''(x_1)\right)(x - x_0),$$

where x_1 is some point between x and x_0. Using (5) we get $F''(x_1) + \lambda''(x_1) \leqslant n \cdot C \cdot C_7 Q + C$. Therefore

$$\left|\left(F''(x_1) + \lambda''(x_1)\right)(x - x_0)\right| \leqslant (n \cdot C \cdot C_7 Q + C) \cdot 2C_3 Q^{-n-1} \leqslant \frac{1}{2} Q - C.$$

Finally we get that for all real x such that $|x - x_0| \leqslant 2C_3 Q^{-n-1}$ the following inequality is satisfied

$$\left|F'(x) + \lambda'(x)\right| \geqslant \left|F'(x_0)\right| - \left|\lambda'(x_0)\right| - \left|\left(F''(x_1) + \lambda''(x)\right)(x - x_0)\right|$$
$$> \left|F'(x_0)\right|/2.$$

In particular this means that the function $F'(x)+\lambda'(x)$ has the same sign within the given interval. Again, on using the Mean Value Theorem we get that $F(x)+\lambda(x)=F(x_0)+\lambda(x_0)+(F'(x_2)+\lambda'(x_2))(x-x_0)$, where x_2 lies between x and x_0. Set $x=x_0\pm 2C_3q^{-n-1}$. Then

$$\begin{aligned}\bigl|\bigl(F'(x_2)+\lambda'(x_2)\bigr)(x-x_0)\bigr| &> 2C_3Q^{-n-1}\bigl|F'(x_0)\bigr|/2\\ &\geqslant C_3Q^{-n}\geqslant \bigl|F(x_0)+\lambda(x_0)\bigr|.\end{aligned}$$

Note that for the two different values of x the expression

$$\bigl(F'(x_2)+\lambda'(x_2)\bigr)\cdot(x-x_0)$$

has different signs. Therefore the value of $F(x)+\lambda(x)=F(x_0)+\lambda(x_0)+(F'(x_2)+\lambda'(x_2))(x-x_0)$ has different signs at the two ends of the interval

$$\bigl[x_0-2C_3Q^{-n-1}, x_0+2C_3Q^{-n-1}\bigr].$$

Thus the function $F(x)+\lambda(x)$ has a root within this interval which completes the proof of Lemma 3. □

In view of Lemma 3, we have that for all ξ satisfying system (14) there exists α with $H(\alpha)\leqslant C_7Q$ such that

$$F(\alpha)+\lambda(\alpha)=0$$

and

$$|\xi-\alpha|<2C_3Q^{-n-1}.$$

Finally, for all $\xi\in I\backslash\Phi(C_1Q,\delta)$ we have constructed a function $F\in\mathcal{F}_n$ such that (14) is satisfied. Therefore, by taking $K_1=C_7$ and $K_2=2C_3$, we find a number $\alpha\in R_\lambda$ satisfying (9). This completes the proof of Proposition 1.

3. Upper bounds: proof of Theorem 2

3.1. Preliminary notes

First of all note that, by Corollary 1, it suffices to establish the upper bound

$$\dim A_2(\psi,\lambda)\leqslant\frac{3}{\tau_\psi+1}. \tag{15}$$

Note that there is nothing to prove if $\tau_\psi=2$. Thus, without loss of generality we can assume that $\tau_\psi>2$. Further, the definition of τ_ψ readily implies that for any $v<\tau_\psi$ we have that $\psi(q)\ll q^{-v}$ for all sufficiently large q. It follows that for any $v<\tau_\psi$ we have that $A_2(\psi,\lambda)\subset A_2(v,\lambda)$.

Therefore, (15) will follow if we consider the special case of $\psi(q) = q^{-v}$ with $2 < v < \tau_\psi$ and let $v \to \tau_\psi$. Therefore, from now on we fix a $v > 2$ and concentrate on establishing the bound

$$\dim A_2(v, \lambda) \leqslant \frac{3}{v+1}. \tag{16}$$

3.2. Auxiliary lemmas

As in the proof of Proposition 1, there is no loss of generality in assuming that $f_1(x) = x$. Then we simply denote $f_2(x)$ by $f(x)$. With the aim of establishing Theorem 2 we fix $v > 2$. By the conditions of Theorem 2, we have that $f''(x) \neq 0$ for all x except a set of Hausdorff dimension $\leqslant \frac{3}{v+1}$. Using the standard arguments — see [5] — we can assume without loss of generality that

$$c_1 \leqslant \left|f''(x)\right| \leqslant c_2 \quad \text{for all } x \in I, \tag{17}$$

where c_1, c_2 are positive constants.

Lemma 4. *(See Pyartly [26].) Let $\delta, \nu > 0$ and $I \subset \mathbb{R}$ be some interval. Let $\phi \in C^n(I)$ be a function such that $|\phi^{(n)}(x)| > \delta$ for all $x \in I$. Then there exists a constant $c(n)$ which depends only on n, such that*

$$\left|\left\{x \in I\colon \left|\phi(x)\right| < \nu\right\}\right| \leqslant c(n)\left(\frac{\nu}{\delta}\right)^{\frac{1}{n}}.$$

Before stating the next lemma recall that $\mathcal{F}_2$ is the set of all functions of the form $a_0 + a_1x + a_2 f(x)$, where a_0, a_1 and a_2 are integers not all zero; $H = H(F) = \max\{|a_1|, |a_2|\}$.

Lemma 5. *There are constants $C_1 > 0$ and $\epsilon_0 > 0$ such that for all $F \in \mathcal{F}_2$ and any subinterval $J \subset I$ of length $|J| \leqslant \epsilon_0$ at least one of the following inequalities is satisfied for all $x \in J$:*

$$\left|F'(x) + \lambda'(x)\right| > C_1 H(F) \quad \textit{or} \quad \left|F''(x) + \lambda''(x)\right| > C_1 H(F).$$

Proof. For the case of $\lambda(x) \equiv 0$ this is proved in [5, Lemmas 5, 6]. To finish the proof in the inhomogeneous case it is sufficient to note that $|\lambda'(x)| \ll 1$ and $|\lambda''(x)| \ll 1$. □

In what follows without loss of generality we can assume that $|I| \leqslant \epsilon_0$, see [5] for analogues arguments.

Lemma 6. *Fix some $0 \leqslant \delta \leqslant 1$ and a positive number H. Denote by $N(\delta)$ the number of triples $(a_0, a_1, a_2) \in \mathbb{Z}^3$ satisfying $\max\{|a_i|\colon i = 0, 1, 2\} \leqslant H$ such that there exists a solution $x \in I$ to the system*

$$\begin{cases} \left|F(x) + \lambda(x)\right| \leqslant H^{-v}; \\ \left|F'(x) + \lambda'(x)\right| \leqslant H^{\delta}. \end{cases} \tag{18}$$

Then for $v > 0$, $N(\delta) \ll H^{1+\delta}$.

Proof. Since $|\lambda'(x)| \ll 1$, $|\lambda(x)| \ll 1$ and $\delta \geqslant 0$, we have that (18) implies the following system

$$\begin{cases} |F(x)| \ll H^{\delta}; \\ |F'(x)| \ll H^{\delta}. \end{cases} \tag{19}$$

Subtracting the second inequality of (19) multiplied by x from the first inequality of (19) gives

$$\begin{cases} |a_0 + a_2(f(x) - xf'(x))| \ll H^{\delta}; \\ |a_1 + a_2 f'(x)| \ll H^{\delta}. \end{cases} \tag{20}$$

If $|a_1| = H$ then we have $2H + 1$ possibilities for a_2. By (17), for each fixed pair (a_1, a_2) the interval of x satisfying the second inequality of (20) is of length $O(H^{\delta} a_2^{-1})$. Therefore the range of $a_2(f(x) - xf'(x))$ is $O(H^{\delta})$. Hence, for every fixed pair (a_1, a_2) we have $O(H^{\delta})$ possibilities for a_0. Thus, we have $O(H^{\delta+1})$ triples (a_0, a_1, a_2) with $|a_1| = H$.

Consider the case $|a_2| = H$. Note that by (17), $f'(x)$ is strictly monotonic and finite. Therefore one can change variables by setting $t = -f'(x)$; $f(x) - xf'(x) = g(t)$. Note that, by (17), the variable t belongs to some finite interval J. Furthermore, the function $g(t)$ is bounded, continuously differentiable on J and $|g'(t)| \ll 1$. Therefore, the system (20) transforms to

$$\begin{cases} |a_0 + a_2 g(t)| \ll H^{\delta}; \\ |a_1 - a_2 t| \ll H^{\delta}; \\ t \in J; \end{cases} \quad \Longrightarrow \quad \begin{cases} \left|\dfrac{a_0}{a_2} + g(t)\right| \ll H^{\delta-1}; \\ \left|\dfrac{a_1}{a_2} - t\right| \ll H^{\delta-1}; \\ t \in J. \end{cases}$$

Note that

$$g\left(\frac{a_1}{a_2}\right) = g(t + \Delta) = g(t) + \Delta g'(\xi) = g(t) + O(H^{\delta-1}),$$

where $\Delta = \frac{a_1}{a_2} - t$. Hence all solutions of the system are also solutions of the inequality

$$\left| g\left(\frac{a_1}{a_2}\right) + \frac{a_0}{a_2} \right| \ll H^{\delta-1}.$$

One can easily check that for $|a_2| = H$ the number of integer solutions of this inequality is not greater than $CH^{1+\delta}$ for some constant C. Therefore $N(\delta) \ll H^{1+\delta}$ and the proof is complete. $\square$

Lemma 7. *Consider the plane defined by the equation $Ax + By + Cz = D$ where A, B, C, D are integers with $(A, B, C) = 1$. Then the area S of any triangle on this plane with integer vertices is at least $\frac{1}{2}\sqrt{A^2 + B^2 + C^2}$.*

Proof. Denote by $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ some points on the considered plane not all lying on the same line. Take one more integer point $\mathbf{v}$ somewhere outside the plane. We now calculate the volume V of the tetrahedron $\mathbf{xyzv}$.

On the one hand the volume of every tetrahedron with integer vertices is at least $\frac{1}{6}$. Therefore $V \geqslant \frac{1}{6}$.

On the other hand, $V = \frac{1}{3}Sh$, where S is the area of the triangle **xyz** and h is the distance between **v** and the plane. Therefore,

$$\frac{1}{6} \leqslant V = \frac{1}{3}Sh \quad \Longleftrightarrow \quad \frac{2}{h} \leqslant S.$$

Let $\mathbf{v} = (\alpha, \beta, \gamma)$. Then

$$h = \frac{|A\alpha + B\beta + C\gamma - D|}{\sqrt{A^2 + B^2 + C^2}} \geqslant \frac{1}{\sqrt{A^2 + B^2 + C^2}},$$

since α, β and γ are integers and $h > 0$. Thus, $S \geqslant \frac{1}{2}\sqrt{A^2 + B^2 + C^2}$ as required. $\square$

3.3. *Proof of Theorem 2*

Let $\sigma := \frac{3}{v+1}$ be the required bound in (16). The strategy of the proof is to construct a collection of coverings $D_i = \{d_{ij}\colon\ j \in J\}$ of $A_2(v, \lambda)$ by intervals d_{ij} such that for any $\epsilon > 0$

$$\sum_{j \in J} |d_{ij}|^{\sigma+\epsilon} \to 0 \quad \text{as } i \to \infty.$$

The bound (16) will then follow from the definition of Hausdorff dimension. Note that $A_2(v, \lambda)$ can be represented in one of the following forms

$$A_2(v, \lambda) = \bigcap_{n=1}^{\infty} \bigcup_{H=n}^{\infty} \bigcup_{\max\{|a_1|,|a_2|\}=H} A(a_0, a_1, a_2) \quad \text{and}$$

$$A_2(v, \lambda) = \bigcap_{n=1}^{\infty} \bigcup_{t=n}^{\infty} B(t), \tag{21}$$

where $A(a_0, a_1, a_2)$ is the set of $x \in I$ satisfying

$$\bigl|a_0 + a_1 x + a_2 f(x) + \lambda(x)\bigr| < H^{-v} \tag{22}$$

for the particular triple (a_0, a_1, a_2) and

$$B(t) = \bigcup_{2^{t-1} \leqslant H < 2^t} \bigcup_{\max\{|a_1|,|a_2|\}=H} A(a_0, a_1, a_2).$$

Therefore for any $n \in \mathbb{N}$ the collection of sets $A(a_0, a_1, a_2)$ with $H \geqslant n$ is a covering of $A_2(v, \lambda)$. Analogously for any $n \in \mathbb{N}$ the collection of $B(t)$ with $t \geqslant n$ is a covering of $A_2(v, \lambda)$.

Fix some positive small number ϵ. Divide every set $A(a_0, a_1, a_2)$ into three subsets:

$$A_1(a_0,a_1,a_2)=\{x\in A(a_0,a_1,a_2)\colon\ |F'(x)+\lambda'(x)|>H^{1-\epsilon}\};\tag{23}$$

$$A_2(a_0,a_1,a_2)=\{x\in A(a_0,a_1,a_2)\colon\ H^{\frac{2-v}{3}}<|F'(x)+\lambda'(x)|\leqslant H^{1-\epsilon}\};\tag{24}$$

$$A_3(a_0,a_1,a_2)=\{x\in A(a_0,a_1,a_2)\colon\ |F'(x)+\lambda'(x)|\leqslant H^{\frac{2-v}{3}}\}\tag{25}$$

where $F(x)=a_0+a_1x+a_2f(x)$.

For any of these collections we can construct the associated sets $A_2^{(1)}(v,\lambda)$, $A_2^{(2)}(v,\lambda)$ and $A_2^{(3)}(v,\lambda)$ analogously to $A_2(v,\lambda)$ — see (21). One can easily check that

$$A_2(v,\lambda)=A_2^{(1)}(v,\lambda)\cup A_2^{(2)}(v,\lambda)\cup A_2^{(3)}(v,\lambda).$$

Therefore it is sufficient to prove (16) for each of these subsets in turn.

3.3.1. The set $A_2^{(1)}(v,\lambda)$

Since $|\lambda'(x)|\ll 1$, we have that

$$|a_1+a_2f'(x)+\lambda'(x)|>H^{1-\epsilon}\quad\Longrightarrow\quad |a_1+a_2f'(x)|\gg H^{1-\epsilon}.$$

Since $|f''(x)|>d$ for all $x\in I$, we have that $a_1+a_2f'(x)$ is a monotonic function. Therefore the set of $x\in I$ such that $|a_1+a_2f'(x)|\gg H^{1-\epsilon}$ is a union of at most two intervals. For one interval we have that

$$a_1+a_2f'(x)\ll -H^{1-\epsilon}$$

and for the other we have that

$$a_1+a_2f'(x)\gg H^{1-\epsilon}.$$

We see that the sign of $F'(x)+\lambda'(x)$ on each of these intervals doesn't change. Therefore $F(x)+\lambda(x)$ is monotonic on them, where $F(x)=a_0+a_1x+a_2f(x)$.

Thus for sufficiently large H the set $A_1(a_0,a_1,a_2)$ is a union of at most 2 intervals (note that it can be empty, i.e. be a union of empty intervals).

Using Lemma 4 and the inequality in (23) we get that the length of each interval is $\ll H^{-v-1+\epsilon}$.

We will use the following cover of $A_2^{(1)}(v,\lambda)$:

$$C_n=\bigcup_{H=n}^{\infty}A_1(a_0,a_1,a_2).$$

Note that for a fixed H the number of different pairs (a_1,a_2) is no greater than $4H$. By (22) there are $O(H)$ possibilities for a_0 if (a_1,a_2) are fixed. Therefore an appropriate s-volume sum for C_n will be

$$\mathcal{H}^s(C_n)\ll\sum_{H=n}^{\infty}H^2\cdot H^{s(\epsilon-1-v)}=\sum_{H=n}^{\infty}H^{2-s(1+v-\epsilon)}.$$

This sum tends to zero as $n \to \infty$ if $2 - s(1 + v - \epsilon) < -1$, that is $s > \frac{3}{1+v-\epsilon}$. Thus,

$$\dim\bigl(A_2^{(1)}(v,\lambda)\bigr) \leqslant \frac{3}{1+v-\epsilon}. \tag{26}$$

3.3.2. The set $A_2^{(2)}(v,\lambda)$

Here we have the inequality $|F'(x) + \lambda'(x)| \leqslant H^{1-\epsilon}$. Therefore Lemma 5 implies

$$\forall x \in A_2(a_0, a_1, a_2), \quad \bigl|F''(x) + \lambda''(x)\bigr| \gg H. \tag{27}$$

In other words $|a_2 f''(x) + \lambda''(x)| \gg H$. This implies that $|a_2| \gg H$. Note that (27) is also true in the case of $x \in A_3(a_0, a_1, a_2)$.

Let δ be an arbitrary number in (0, 1]. Consider the set

$$A_\delta(v,\lambda) = \bigcap_{n=1}^{\infty} \bigcup_{H=n}^{\infty} A_\delta(a_0, a_1, a_2), \tag{28}$$

where $A_\delta(a_0, a_1, a_2)$ is the set of $x \in A_2(a_0, a_1, a_2)$ with the following property

$$H^{1-\frac{1}{3}(v+1)\delta} < \bigl|F'(x) + \lambda'(x)\bigr| \leqslant H^{1-\delta}. \tag{29}$$

We have that $|F''(x) + \lambda''(x)| \gg H$. Therefore, the set $A_\delta(a_0, a_1, a_2)$ consists of at most 4 intervals. Consider the following cover C_n for $A_\delta(v,\lambda)$:

$$C_n = \bigcup_{H=n}^{\infty} A_\delta(a_0, a_1, a_2).$$

By Lemma 6, for a fixed H there exist only $O(H^{2-\delta})$ non-empty sets $A_\delta(a_0, a_1, a_2)$. By Lemma 4 the length of each interval in $A_\delta(a_0, a_1, a_2)$ is bounded by $H^{-v-1+\frac{1}{3}(v+1)\delta}$. Therefore the corresponding s-volume sum for C_n is bounded by

$$\sum_{H=n}^{\infty} H^{2-\delta} \cdot H^{s(-v-1+\frac{1}{3}(v+1)\delta)}. \tag{30}$$

If $s > \frac{3}{v+1}$ then the exponent of H is equal to

$$2 - \delta + s\left(-v - 1 + \frac{1}{3}(v+1)\delta\right) < 2 - \delta - 3 + \delta = -1.$$

Hence for $s > \frac{3}{v+1}$ the right hand side of (30) tends to 0 as $n \to \infty$. It follows that $\dim(A_\delta(v,\lambda)) \leqslant \frac{3}{v+1}$ for any $\delta \in [0,1]$.

For simplicity denote by k the quantity $\frac{1}{3}(v+1)$. Note that the set $A_2^{(2)}(v,\lambda)$ can be expressed as the finite union

$$A_2^{(2)}(v,\lambda)=\bigcup_{i=1}^{l} A_{\delta_i}(v,\lambda)\cup A_{\delta^*}(v,\lambda), \tag{31}$$

where $\delta_1=\epsilon$, $\delta_{i+1}=k\delta_i$, $\delta^*=1$. Since $k>1$ we have that $\delta_i\to\infty$ as $i\to\infty$. Therefore there exists a natural number l which depends on ϵ only such that $\delta_{l+1}>1$ and $\delta_l\leqslant 1$.

Since the Hausdorff dimension of each set $A_{\delta_i}(v,\lambda)$ and $A_{\delta^*}(v,\lambda)$ appearing in (31) is not greater than $\frac{3}{v+1}$, we get that $\dim(A_2^{(2)}(v,\lambda))\leqslant\frac{3}{v+1}$.

3.3.3. The set $A_2^{(3)}(v,\lambda)$

Consider the set

$$B_3(t):=\bigcup_{2^{t-1}\leqslant H<2^t} A_3(a_0,a_1,a_2)$$

and let $\delta:=\frac{2-v}{3}$.

Recall that for all $x\in A_3(a_0,a_1,a_2)$ we have $|F''(x)+\lambda''(x)|\gg H$. Therefore, by Lemma 4, we have that the length of each interval in $A_3(a_0,a_1,a_2)$ with $H\asymp 2^t$ is not greater than

$$\left(H^{-v}/H\right)^{\frac{1}{2}}\ll 2^{-t(\frac{v+1}{2})}.$$

Fix a sufficiently small positive number ϵ_1. Let $c=1+\epsilon_1$. For every t divide the interval I into 2^{ct} equal subintervals of length $2^{-ct}|I|\ll 2^{-ct}$. These subintervals are divided into two classes:

- Class I intervals. They include at most $O(2^{t(\frac{3}{2}-c)})$ segments from $B_3(t)$.
- Class II intervals. They include those which are not in class I.

According to this classification consider the sets

$$A_{\mathrm{I}}(v,\lambda)=\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\bigcup_{\text{class I intervals } J} B_3(t)\cap J;$$

$$A_{\mathrm{II}}(v,\lambda)=\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\bigcup_{\text{class II intervals } J} B_3(t)\cap J.$$

It follows that

$$A_2^{(3)}(v,\lambda)=A_{\mathrm{I}}(v,\lambda)\cup A_{\mathrm{II}}(v,\lambda).$$

The required upper bound for $A_{\mathrm{I}}(v,\lambda)$ will follow on showing the following lemma.

Lemma 8. $\dim(A_{\mathrm{I}}(v,\lambda))\leqslant\frac{3}{v+1}(1+\epsilon_1)$.

Proof. Consider a class I interval J. We have at most $O(2^{t(\frac{3}{2}-c)})$ segments from $B_3(t)$ on it. Therefore there are not greater than $O(2^{\frac{3}{2}t})$ intervals from $B_3(t)$ lying inside class I intervals. Consider the following cover of $A_{\mathrm{I}}(v,\lambda)$:

$$C_n := \bigcup_{t=n}^{\infty} \bigcup_{\text{class I intervals } J} B_3(t) \cap J.$$

Its $\frac{3}{v+1}(1+\epsilon_1)$-volume is bounded by

$$\sum_{t=n}^{\infty} 2^{\frac{3}{2}t} \cdot 2^{-t(\frac{v+1}{2})\cdot\frac{3(\epsilon_1+1)}{v+1}} = \sum_{t=n}^{\infty} 2^{(\frac{3}{2}-\frac{3}{2}(\epsilon_1+1))t} = \sum_{t=n}^{\infty} 2^{-\frac{3}{2}\epsilon_1 t}.$$

It obviously tends to zero as $n \to \infty$. This finishes the proof of the lemma. □

Let J be a class II interval and $F(x) = a_0 + a_1 x + a_2 f(x) \in \mathcal{F}_2$ with $2^{t-1} \leqslant H(F) < 2^t$ and $A_3(a_0,a_1,a_2) \cap J \neq \emptyset$. Then

$$\left|F(x_0)+\lambda(x_0)\right| \ll 2^{-vt} \quad \text{and} \quad \left|F'(x_0)+\lambda'(x_0)\right| \ll 2^{\delta t}$$

for some $F \in \mathcal{F}_2$ and $x_0 \in J$. Then for all $x \in J$ we have

$$\left|F'(x)+\lambda'(x)\right| = \left|(F'+\lambda')(x_0) + (x-x_0)(F''+\lambda'')(\xi)\right| \ll 2^{\delta t} + 2^{(1-c)t},$$

$$\left|F(x)+\lambda(x)\right| = \left|(F+\lambda)(x_0) + (x-x_0)(F'+\lambda')(x_0) + \frac{(x-x_0)^2}{2}(F''+\lambda'')(\xi)\right|$$
$$\ll 2^{-vt} + 2^{(\delta-c)t} + 2^{(1-2c)t}.$$

Choose $\epsilon_1 > 0$ sufficiently small such that

$$v > 2 + 3\epsilon_1. \tag{32}$$

Then we have

$$2^{\delta t} < 2^{(1-c)t} \quad \text{and} \quad 2^{(\delta-c)t} < 2^{(1-2c)t}.$$

One can see that 2^{-vt} is always less than the other summands $2^{(\delta-c)t}$ and $2^{(1-2c)t}$. Hence in the case of (32) we get the inequalities

$$\left|F(x)+\lambda(x)\right| \ll 2^{(1-2c)t}, \tag{33}$$

$$\left|F'(x)+\lambda'(x)\right| \ll 2^{(1-c)t} \tag{34}$$

for all $x \in J$.

Lemma 9. *For every fixed J as above all points $\vec{a} = (a_0,a_1,a_2) \in \mathbb{Z}^3$ such that $A_3(a_0,a_1,a_2) \cap J \neq \emptyset$ lie on a single affine plane.*

Proof. Suppose that there exist four integer points $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ not lying on the same plane such that $A_3(\vec{a}) \cap J \neq \emptyset$, $A_3(\vec{b}) \cap J \neq \emptyset$, $A_3(\vec{c}) \cap J \neq \emptyset$ and $A_3(\vec{d}) \cap J \neq \emptyset$. It means that the points $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ form a tetrahedron with integer vertexes. Therefore its volume is at least $\frac{1}{6}$.

On the other hand all of these four points must lie inside a parallelepiped R formed by the inequalities (33), (34) and $|a_2| < H$ for a fixed $x \in J$. The volume of this figure is bounded by

$$V \ll 2 \cdot 2^{t(1-2c)} \cdot 2 \cdot 2^{t(1-c)} \cdot 2 \cdot 2^t \cdot D^{-1} \ll 2^{t(3-3c)} \cdot D^{-1},$$

where D is the determinant of the matrix

$$\begin{pmatrix} 1 & x & f(x) \\ 0 & 1 & f'(x) \\ 0 & 0 & 1 \end{pmatrix}$$

i.e. $D = 1$. Since $c > 1$ we have $V = o(1)$ contrary to $V \geqslant 1/6$. The proof is complete. $\square$

Let the plane from Lemma 9 have the form $Ax + By + Cz = D$. We evaluate the intersection area of this plane with parallelepiped R specified in the proof of the lemma. In order to do this let us consider the body P_Δ given by the inequalities

$$\begin{cases} \left|F(x) + \lambda(x)\right| \leqslant 2^{t(1-2c)}; \\ |a_2| \leqslant 2^t; \\ |Aa_0 + Ba_1 + Ca_2 - D| \leqslant \Delta, \end{cases} \tag{35}$$

where $\Delta > 0$ is a positive parameter. Here a_0, a_1, a_2 are viewed as real variables. The volume of P_Δ can be expressed in two different ways. Firstly, since the determinant of system (35) is $B - Ax$, we have that

$$V(P_\Delta) = \frac{8 \cdot 2^{t(2-2c)} \cdot \Delta}{|B - Ax|}. \tag{36}$$

Secondly, P_Δ is a parallelepiped so its volume is $V(P_\Delta) = S \cdot h$ where S is the area of the faces defined by the third inequality of (35) and h is a distance between these faces. That is

$$V(P_\Delta) = S \cdot \frac{2\Delta}{\sqrt{A^2 + B^2 + C^2}}. \tag{37}$$

Hence on combining (36) and (37) we obtain that

$$S \asymp \frac{2^{t(2-2c)} \cdot \sqrt{A^2 + B^2 + C^2}}{|B - Ax|}. \tag{38}$$

Note that S is the area of the intersection of the plane $Aa_0 + Ba_1 + Ca_2 - D = 0$ with the figure defined by the first two inequalities of (35). Therefore the intersection area of this plane with the parallelepiped is not greater than S. Note that all points $\mathbf{a}$ should lie inside this intersection and (38) gives an estimate for its area.

Case (i): We consider intervals J of type II such that not all points $\mathbf{a}$ associated with J lie on the same line. By Lemma 7, we get that the number N of such points on a fixed interval J is bounded by

$$N \ll \frac{2^{t(2-2c)} \cdot \sqrt{A^2+B^2+C^2}}{|B-Ax|} / \sqrt{A^2+B^2+C^2} = \frac{2^{t(2-2c)}}{|B-Ax|}. \tag{39}$$

Since J is not a class I interval we get that for all $x \in J$,

$$|B-Ax| \ll 2^{t(\frac{1}{2}-c)} = 2^{-t(\frac{1}{2}+\epsilon_1)}. \tag{40}$$

Similarly to (35) we consider two more systems of inequalities:

$$\begin{cases} \left|F(x)+\lambda(x)\right| \leqslant 2^{t(1-2c)}; \\ |Aa_0+Ba_1+Ca_2-D| \leqslant \Delta; \\ \left|F'(x)+\lambda'(x)\right| \leqslant 2^{t(1-c)} \end{cases} \quad \text{and} \quad \begin{cases} \left|F'(x)+\lambda'(x)\right| \leqslant 2^{t(1-c)}; \\ |a_2| \leqslant 2^t; \\ |Aa_0+Ba_1+Ca_2-D| \leqslant \Delta. \end{cases}$$

Analogously we get additional bounds for N, namely

$$N \ll \frac{2^{t(2-3c)}}{|T|} \quad \text{and} \quad N \ll \frac{2^{t(2-c)}}{|A|}, \tag{41}$$

where

$$T = \det\begin{pmatrix} 1 & x & f(x) \\ A & B & C \\ 0 & 1 & f'(x) \end{pmatrix} = f'(x)(B-Ax) - \bigl(C - Af(x)\bigr).$$

Since J is a class II interval then

$$\left|f'(x)(B-Ax) - \bigl(C-Af(x)\bigr)\right| \ll 2^{t(\frac{1}{2}-2c)}.$$

This result with (40) implies

$$\left|C - Af(x)\right| \ll 2^{t(\frac{1}{2}-c)} = 2^{-t(\frac{1}{2}+\epsilon_1)}. \tag{42}$$

The second inequality in (41) together with the fact that J is a class II interval implies $|A| \ll 2^{\frac{1}{2}t}$.

Fix A. Denote by $M(A)$ the number of possible integer triples (A, B, C) which can be the coefficients of a plane corresponding to some class II interval. As we have shown such a triple should satisfy (40) and (42) for some $x \in I$. By (40) the number of possible parameters B is the number of fractions $\frac{B}{A}$ in I which is bounded by $|I||A|+1$. Inequality (40) also implies that for fixed B the value Ax can only lie in some interval L of length

$$|L| \ll 2^{-t(\frac{1}{2}+\epsilon)}.$$

Since the range for the value $Af(x)$ is comparable to that for Ax then by (42) for fixed A and B the number of possibilities for C is $O(1)$. Therefore finally we get an estimate $M(A) \ll |A|$.

Suppose there exist two class II intervals J_1 and J_2 with the same coefficients (A, B, C) of the appropriate plane. Applying inequality (40) we get

$$\begin{cases} \forall x \in J_1, & |B - Ax| \ll 2^{t(\frac{1}{2}-c)}; \\ \forall y \in J_2, & |B - Ay| \ll 2^{t(\frac{1}{2}-c)}; \end{cases} \quad \Rightarrow \quad \bigl|A(x-y)\bigr| \ll 2^{t(\frac{1}{2}-c)} \Rightarrow |x-y| \ll \frac{2^{t(\frac{1}{2}-c)}}{|A|}.$$

Therefore for a fixed (A, B, C) the number x can only lie in an interval of length $\frac{2^{t(\frac{1}{2}-c)}}{|A|}$. Since the length of each class II interval is 2^{-ct} then the number of class II intervals associated with the triple (A, B, C) is at most $\frac{2^{\frac{1}{2}t}}{|A|}$.

We will use the following cover for $A_{\mathrm{II}}(v, \lambda)$:

$$C_n = \bigcup_{t=n}^{\infty} \bigcup_{J \text{ are class II intervals}} B_3(t) \cap J.$$

Using the second inequality in (41) to estimate the number of intervals in $B_3(t) \cap J$ we get that the s-volume sum for this cover is bounded by

$$\mathcal{H}^s(C_n) \leqslant \sum_{t=n}^{\infty} \sum_{(A,B,C)} \frac{2^{\frac{1}{2}t}}{|A|} \cdot \frac{2^{t(2-c)}}{|A|} \cdot 2^{-st(\frac{v+1}{2})} = \sum_{t=n}^{\infty} 2^{t(\frac{3}{2}-\epsilon_1-s(\frac{v+1}{2}))} \sum_{(A,B,C)} \frac{1}{|A|^2},$$

where (A, B, C) run through all possible coefficients of planes corresponding to the type II intervals under consideration. Transforming this series we get

$$\begin{aligned} \sum_{t=n}^{\infty} 2^{t(\frac{3}{2}-\epsilon_1-s(\frac{v+1}{2}))} \sum_{|A|=1}^{2^{\frac{t}{2}}} \frac{1}{|A|} &\ll \sum_{t=n}^{\infty} t \cdot 2^{t(\frac{3}{2}-\epsilon_1-s(\frac{v+1}{2}))} \\ &\ll \sum_{t=n}^{\infty} 2^{t(\frac{3}{2}-s(\frac{v+1}{2}))}. \end{aligned} \tag{43}$$

If $s > \frac{3}{v+1}$ then this series obviously tends to zero as $n \to \infty$.

Case (ii): We consider intervals J of type II such that all points $\mathbf{a}$ associated with J lie on the same line L. Fix such an interval J. Represent this line in the form:

$$\alpha + t\beta$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ is an integer point on L, $\beta = (\beta_0, \beta_1\, \beta_2)$ is the vector connecting the nearest integer points on L and t is an arbitrary real number. Then all the vectors (a_0, a_1, a_2) associated with J are of the form

$$a_0 = \alpha_0 + k\beta_0, a_1 = \alpha_1 + k\beta_1, a_2 = \alpha_2 + k\beta_2,$$

where α_i, β_i are fixed and $k \in \mathbb{Z}$ varies. Since J is of class II, there are at least $2^{t(\frac{3}{2}-c)}$ different values k so there exist k_1 and k_2 such that $|k_1 - k_2| \geqslant 2^{t(3/2-c)}$. For each vector (a_0, a_1, a_2) under consideration we have that $|a_0| \ll 2^t$. Hence taking values of $|a_0|$ for two different vectors for J and subtracting one value from another we get

$$\bigl|\beta_0(k_1 - k_2)\bigr| \ll 2^t \quad \Rightarrow \quad |\beta_0| \leqslant 2^{t(c-\frac{1}{2})}. \tag{44}$$

Similarly we obtain the same inequalities for β_1 and β_2.

Now consider inequalities (33) and (34) for the same two vectors. Again subtracting one inequality from the other we get

$$\begin{cases} \bigl|(k_1 - k_2)\bigl(\beta_0 + \beta_1 x + \beta_2 f(x)\bigr)\bigr| \leqslant 2 \cdot 2^{t(1-2c)}; \\ \bigl|(k_1 - k_2)\bigl(\beta_1 + \beta_2 f'(x)\bigr)\bigr| \leqslant 2 \cdot 2^{t(1-c)}. \end{cases} \tag{45}$$

Since there are at least $2^{t(\frac{3}{2}-c)}$ different values k, we can ensure that $|k_1 - k_2| \geqslant 2^{t(\frac{3}{2}-c)}$ for some k_1, k_2. Dividing these inequalities by $|k_1 - k_2|$ and changing variables as in Lemma 6 gives the system

$$\begin{cases} \bigl|\beta_0 + \beta_2 g(y)\bigr| \leqslant 2 \cdot 2^{-\frac{t}{2}}; \\ |\beta_1 + \beta_2 y| \leqslant 2 \cdot 2^{-\frac{t}{2}} \end{cases}$$

where $y = f'(x)$ and $g(y) = f(x) - xf'(x)$. Using (44) we get $H = \max\{|\beta_0|, |\beta_1|, |\beta_2|\} \ll 2^{t(c-\frac{1}{2})}$. Substituting this into the system we get

$$\begin{cases} \bigl|\beta_0 + \beta_2 g(y)\bigr| \ll H^{\frac{1}{1-2c}} = H^{-\frac{1}{1+2\epsilon_1}}; \\ |\beta_1 + \beta_2 y| \ll H^{\frac{1}{1-2c}} = H^{-\frac{1}{1+2\epsilon_1}}. \end{cases} \tag{46}$$

For a fixed value of β_2 the number of possibilities for β_1 is $O(|\beta_2|)$. For a fixed β_1 and β_2 the number of possibilities for β_0 is $O(1)$. Denote by $K(\beta_2)$ the number of solutions $(\beta_0, \beta_1, \beta_2)$ of (46), where β_2 is fixed. Then we get that $K(\beta_2) \ll |\beta_2|$.

Suppose that for two different intervals J_1 and J_2 the parameters β_0, β_1 and β_2 coincide. Using the second inequality in (46) we get

$$\bigl|\beta_2(y_1 - y_2)\bigr| \ll H^{-\frac{1}{1+2\epsilon_1}}$$

where $y_1 \in f'(J_1)$ and $y_2 \in f'(J_2)$. Since for all $x \in I$, $f'(x) > d > 0$ the inequality can be transformed to the form

$$|x_1 - x_2| \ll \frac{H^{-\frac{1}{1+2\epsilon_1}}}{|\beta_2|} \ll \frac{2^{-\frac{1}{2}t}}{|\beta_2|}.$$

Therefore the number of class II intervals J with parameters $\beta_0, \beta_1, \beta_2$ is not greater than

$$\frac{2^{t(c-\frac{1}{2})}}{|\beta_2|}. \tag{47}$$

Further, since $|\alpha_2 + k\beta_2| \leqslant 2^t$ there are at most $\frac{2^t}{|\beta_2|}$ intervals inside $J \cap B_3(t)$. Using (47) we have the following upper bound for the s-volume sum:

$$\begin{aligned}\mathcal{H}^s(C_n) &= \sum_{t=n}^{\infty} \sum_{(\beta_0,\beta_1,\beta_2)} \frac{2^{t(c-\frac{1}{2})}}{|\beta_2|} \cdot \frac{2^t}{|\beta_2|} \cdot 2^{-st(\frac{v+1}{2})} = \sum_{t=n}^{\infty} 2^{t(\frac{3}{2}+\epsilon_1 - s(\frac{v+1}{2}))} \sum_{(\beta_0,\beta_1,\beta_2)} \frac{1}{|\beta_2|^2} \\ &\ll \sum_{t=n}^{\infty} 2^{t(\frac{3}{2}+\epsilon_1 - s(\frac{v+1}{2}))} \sum_{|\beta_2| \leqslant 2^{t(c-\frac{1}{2})}} \frac{K(\beta_2)}{|\beta_2|^2} \\ &\ll \sum_{t=n}^{\infty} t \cdot 2^{t(\frac{3}{2}+\epsilon_1 - s(\frac{v+1}{2}))}.\end{aligned}$$

Using the same arguments as in the case when (a_0, a_1, a_2) lie on a plane we obtain that

$$\mathcal{H}^s(C_n) \ll \sum_{t=n}^{\infty} 2^{t(\frac{3}{2}+2\epsilon_1 - s(\frac{v+1}{2}))}.$$

Combining this series with (43) we get an estimate

$$\dim\bigl(A_{\mathrm{II}}(v,\lambda)\bigr) \leqslant \frac{3+4\epsilon_1}{v+1}.$$

Therefore finally for $v > 2 + 3\epsilon_1$ we get that

$$\begin{aligned}\dim\bigl(A_2(v,\lambda)\bigr) &= \dim\bigl(A_2^{(1)}(v,\lambda) \cup A_2^{(2)}(v,\lambda) \cup A_{\mathrm{I}}(v,\lambda) \cup A_{\mathrm{II}}(v,\lambda)\bigr) \\ &\leqslant \max\left\{\frac{3}{1+v-\epsilon}, \frac{3+4\epsilon_1}{v+1}, \frac{3}{v+1}(1+\epsilon_1)\right\}.\end{aligned}$$

Since ϵ and ϵ_1 can be made arbitrary small then all values in the maximum can be made arbitrary close to $\frac{3}{v+1}$, thus implying (15) and thereby completing the proof of Theorem 2.

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